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Initial Lagrange Multipliers for the Shooting Method

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I. Introduction

THE purpose of this Note is to concisely discuss the analytical aspects of a number of methods for obtaining initial Lagrange multipliers for the shooting method. Because the ultimate goal is to use the shooting method, the partial derivatives of the state equation with respect to the time, the state, and the control are available and can be used to enhance the performance of these methods. The methods considered are parameterizing the control within the shooting method, direct shooting, collocation (direct transcription), pseudospectral method, dynamic programming, and adjoint-control transformation. The intent is to discuss each of these methods briefly to present their important features.

A standard optimal control problem without path constraints is stated, as are the corresponding optimality conditions. Next, a brief derivation of the shooting method is presented to show what information would be available for initial multiplier prediction and to show how to parameterize the control within the shooting method. Then the other methods are discussed briefly.

II. Optimal Control Problem/Optimality Conditions

The optimal control problem considered here is finding the control history $u(t)$ that minimizes the performance index

$$J = \phi(t_f, x_f) \tag{1}$$

subject to the differential constraints

$$\dot{x} = f(t, x, u) \tag{2}$$

and the prescribed boundary conditions

$$t_0 = 0, \qquad x_0 = x_{0_s}, \qquad \psi(t_f, x_f) = 0 \tag{3}$$

where u, x, and ψ are $m \times 1$, $n \times 1$, and $p \times 1$. This is a well-known problem, and the details can be found in [1].

For the discussion that follows, it is convenient to normalize the final time as $\tau = t/t_f$, which makes t_f a parameter. The optimal control problem becomes the following:

$$\begin{gathered} J = \phi(x_f, t_f), \qquad x' = t_f f(t_f \tau, x, u) \stackrel{\Delta}{=} g(\tau, x, u, t_f) \\ \tau_0 = 0, \qquad x_0 = x_{0_s}, \qquad \tau_f = 1, \qquad \psi(x_f, t_f) = 0 \end{gathered} \tag{4}$$

where the prime denotes a derivative with respect to τ. This is now a *fixed-final-time* problem.

Next, the final time can be made a state $y \stackrel{\Delta}{=} t_f$ by including the differential constraint $y' = 0$. The optimal control problem becomes

$$\begin{gathered} J = \phi(x_f, y_f) \qquad x' = g(\tau, x, y, u), \qquad y' = 0 \\ \tau_0 = 0, \qquad x_0 = x_{0_s}, \qquad \tau_f = 1, \qquad \psi(x_f, y_f) = 0 \end{gathered} \tag{5}$$

This is now a *free-initial-condition* problem because y_0 is free.

The constraints are adjoined to the performance index by multipliers $\lambda(\tau)$, $\mu(\tau)$, and ν to form the augmented performance index:

$$J' = G(x_f, y_f, \nu) + \int_{t_0}^{t_f} [H(\tau, x, y, u, \lambda) - \lambda^T x' - \mu^T y'] \, \mathrm{d}t \tag{6}$$

where, because $y' = 0$,

$$G = \phi + \nu^T \psi, \qquad H = \lambda^T g \tag{7}$$

The optimality conditions for this problem are given by

$$\begin{gathered} x' = g, \qquad y' = 0, \qquad \lambda' = -H_x^T, \qquad \mu' = -H_y^T, \qquad H_u^T = 0 \\ \tau_0 = 0, \qquad x_0 = x_{0_s}, \qquad \mu_0 = -G_{y_0}^T = 0 \\ \tau_f = 1, \qquad \psi = 0, \qquad \lambda_f = G_{x_f}^T, \qquad \mu_f = G_{y_f}^T \end{gathered} \tag{8}$$

Note that $y = t_f$ is a scalar in the stated problem, but it is treated as an $r \times 1$ vector of parameter states for later use.

III. Shooting Method

It is assumed that $H_u^T = 0$ can be solved for the control as $u = u(\tau, x, y, \lambda)$ so that the following two-point boundary-value problem (TPBVP) can be formed:

$$z' = F(\tau, z), \qquad \tau_0 = 0, \qquad x_0 = x_{0_s}, \qquad \mu_0 = 0, \qquad h(z_f) = 0 \tag{9}$$

where

$$z = \begin{bmatrix} x \\ y \\ \lambda \\ \mu \end{bmatrix}, \qquad F = \begin{bmatrix} g(\tau, x, y, u(\tau, x, y, \lambda), \lambda) \\ 0 \\ -H_x^T(\tau, x, y, u(\tau, x, y, \lambda), \lambda) \\ -H_y^T(\tau, x, y, u(\tau, x, y, \lambda), \lambda) \end{bmatrix} \tag{10}$$

The dimension of z is $(n + r + n + r) \times 1$, and the dimension of h is $(n + r) \times 1$. The latter is obtained by eliminating the p multipliers ν from the $p + n + r$ final conditions ($\psi = 0$, $\lambda_f = G_{x_f}^T$, and $\mu_f = G_{y_f}^T$).

For given values of y_0 and λ_0, the differential equation can be integrated to obtain z_f, for which $h = 0$ is not satisfied. Next, Eqs. (9) are linearized about the nominal path to find the changes δy_0 and $\delta\lambda_0$ that drive $\|h\|$ toward zero.

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The linearized z equation is given by

$$\frac{d(\delta z)}{d\tau} = F_z \delta_z \tag{11}$$

because δx_0 and $\delta \mu_0$ are zero, the solution of Eq. (11) is given by

$$\delta z = \Psi(\tau, \tau_0) \begin{bmatrix} \delta y_0 \\ \delta \lambda_0 \end{bmatrix} \tag{12}$$

where Ψ is the $(2n + 2r) \times (r + n)$ transition matrix satisfying the equations

$$\Psi' = F_z \Psi, \qquad \Psi_0 = \begin{bmatrix} 0_{n\times r} & 0_{n\times n} \\ I_{r\times r} & 0_{r\times n} \\ 0_{n\times r} & I_{n\times n} \\ 0_{r\times r} & 0_{r\times n} \end{bmatrix} \tag{13}$$

Hence,

$$\delta z_f = \Psi_f \begin{bmatrix} \delta y_0 \\ \delta \lambda_0 \end{bmatrix} \tag{14}$$

The linearized final condition is given by

$$h_{z_f} \delta_{z_f} = -\alpha h \tag{15}$$

where α is a parameter that controls the reduction in $\| h \|$ on each iteration. This equation can be rewritten as

$$h_{z_f} \Psi_f \begin{bmatrix} \delta y_0 \\ \delta \lambda_0 \end{bmatrix} = -\alpha h \tag{16}$$

which is solved as a linear system for δy_0 and $\delta \lambda_0$.

A problem with the shooting method is that good values are needed for y_0 and λ_0 to make it work; recall that $y_0 = t_f$. The rest of this Note is concerned with getting values of y_0 and λ_0 using suboptimal control methods. Conversion of optimal control problems into suboptimal control problems is discussed in [2].

IV. Shooting Method for Piecewise-Linear Controls

The general idea is to replace the control $u(\tau)$ in the shooting method by a piecewise-linear control: that is, N control nodes $u_1, \ldots, u_N$ at fixed points $\tau_1, \ldots, \tau_N$ plus linear interpolation [3]. In terms of the $r \times 1$ vector,

$$y \triangleq [t_f, u_1, \ldots, u_N]^T \tag{17}$$

Equations (5) become

$$J = \phi(x_f, y_{1_f}) \qquad x' = g(\tau, x, y), \qquad y' = 0$$
$$\tau_0 = 0, \qquad x_0 = x_{0_s}, \qquad \tau_f = 1, \qquad \psi(x_f, y_{1_f}) = 0 \tag{18}$$

Then, in terms of the functions

$$G = \phi(x_f, y_{1_f}) + \nu^T \psi(x_f, y_{1_f}), \qquad H = \lambda^T g(\tau, x, y) \tag{19}$$

the optimality conditions (8) become

$$x' = g, \qquad y' = 0, \qquad \lambda' = -H_x^T, \qquad \mu' = -H_y^T$$
$$\tau_0 = 0, \qquad x_0 = x_{0_s}, \qquad \mu_0 = -G_{y_0}^T = 0$$
$$\tau_f = 1, \qquad \psi = 0, \qquad \lambda_f = G_{x_f}^T, \qquad \mu_f = G_{y_f}^T \tag{20}$$

These equations lead to the following TPBVP:

$$z' = F(\tau, z), \qquad \tau_0 = 0, \qquad x_0 = x_{0s}, \qquad \mu_0 = 0, \qquad h(z_f) = 0 \tag{21}$$

where

$$z = \begin{bmatrix} x \\ y \\ \lambda \\ \mu \end{bmatrix}, \qquad F = \begin{bmatrix} g(\tau, x, y) \\ 0 \\ -H_x^T(\tau, x, y, \lambda) \\ -H_y^T(\tau, x, y, \lambda) \end{bmatrix} \tag{22}$$

Note that the state equation no longer depends on λ, and so the sensitivity of the TPBVP to λ_0 completely disappears! The remaining discussion of the TPBVP is identical with that associated with Eqs. (11–16). Upon convergence, the resulting y_1 and λ_0 can be used to start the regular shooting method.

V. Direct Shooting

The optimal control problem in Eq. (4) is converted into a parameter optimization problem by replacing the control history $u(\tau)$ by a piecewise-linear control: that is, a set of control nodes $u_1, \ldots, u_N$ (parameters) at the fixed points $\tau_1, \ldots, \tau_N$. Linear interpolation is used to form the function $u(\tau)$. For given values of the $r = 1 + N$ parameters $y = [t_f, u_1, \ldots, u_N]^T$, the state equation is solved by explicit Runge–Kutta integration to obtain $x_f = x_f(y)$. Finally, x_f is eliminated from ϕ and ψ to form the parameter optimization problem of finding the parameters y that minimize the performance index

$$J = \bar{\phi}(y) \tag{23}$$

subject to the equality constraints

$$\bar{\psi}(y) = 0 \tag{24}$$

There are many nonlinear programming codes available for solving this problem; most are based on sequential quadratic programming.

The converged values of the Lagrange multipliers associated with the constraints $\bar{\psi}$ are the ν values of the optimal control problem. Then, with the converged values of t_f and x_f, the final value of the multiplier λ becomes [1]

$$\lambda_f = G_{x_f}^T(t_f, x_f, \nu) \tag{25}$$

Finally, with the optimal piecewise-linear control, the differential equations [1]

$$\dot{x} = f(t, x, u), \qquad \dot{\lambda} = -H_x^T(t, x, u, \lambda) \tag{26}$$

are integrated backward in time to obtain λ_0. If the shooting method does not converge with this t_f and λ_0, more control nodes can be used and the process can be repeated.

To use a nonlinear programming code, it is necessary to have the derivatives of $\bar{\phi}$ and $\bar{\psi}$ in Eqs. (23) and (24) with respect to y. A usual procedure is to use finite differences, which requires many integrations of the system equations. Because the intent is to use the shooting method, the need for deriving the derivative F_z provides the derivatives g_x, g_u, and g_{t_f}. Hence, it is possible to compute the derivatives more accurately by integrating differential equations for transition matrices.

VI. Collocation (Direct Transcription)

For this method of solving the optimal control problem in Eq. (4), controls and states are guessed at the nodes, which now represent integration steps. Then, implicit Runge–Kutta integration is used to define the integration formula for each step; for example, for the second-order trapezoid rule,

$$x_{k+1} = x_k + (h/2)[g(\tau_k, x_k, u_k, t_f) + g(\tau_{k+1}, x_{k+1}, u_{k+1}, t_f)] \tag{27}$$

Because the x values are guessed, Eq. (27) is not satisfied. It is forced to be satisfied by defining the residual for each step to be

$$R_k = x_{k+1} - x_k - (h/2)[g(\tau_k, x_k, u_k, t_f) + g(\tau_{k+1}, x_{k+1}, u_{k+1}, t_f)] \tag{28}$$

and imposing all of the residuals as constraints in the parameter optimization problem.

The parameter optimization problem is to find $t_f, u_1, \ldots, u_N$, and $x_2, \ldots, x_N$, which minimize the performance index

$$J = \phi(x_N, t_f) \tag{29}$$

subject to the constraints

$$\begin{gathered} \psi(x_N, t_f) = 0 \\ R_1(u_1, u_2, x_1, x_2, t_f) = 0 \\ \vdots \\ R_{N-1}(u_{N-1}, u_N, x_{N-1}, x_N, t_f) = 0 \end{gathered} \tag{30}$$

With this method, the optimization and integration are performed simultaneously.

It has been shown [4] that the multipliers associated with the residuals R_k are local values of $-\lambda(t)$ of the optimal control problem. Hence, the multiplier of R_1 is approximately $-\lambda_0$. If the initial conditions on the states had been adjoined to the performance index with multipliers ξ, the endpoint function and λ_0 would have been

$$G = \phi + \nu^T\psi + \xi^T(x_0 - x_{0_s}), \qquad \lambda_0 = -G_{x_0}^T = -\xi \tag{31}$$

Because the ultimate goal is to use the shooting method, the derivatives g_x, g_u, and g_{t_f} are available. Hence, the derivatives needed for using a nonlinear programming code can be calculated analytically.

VII. Pseudospectral Method

One example of this method is the Legendre pseudospectral method [5]. Here, a fixed number of Legendre–Gauss–Lobatto (LGL) points is placed at $\tau_1, \ldots, \tau_N$ over the interval of integration τ_0 and τ_f, and along with t_f, values for x_k and u_k are guessed at the LGL points. Next, after using Lagrange interpolation on the state nodes, the differential equations $x' = g(\tau, x, u, t_f)$ are required to be satisfied at the LGL points. The resulting conditions are imposed as constraints in the parameter optimization problem. The Lagrange multiplier λ_0 is proportional to the multiplier of $x' = g$ at the initial point.

VIII. Method Based on Dynamic Programming

It is known from dynamic programming that [6]

$$\lambda_0^T = \frac{\partial J^{\text{opt}}}{\partial x_0} \tag{32}$$

An approximate method for obtaining λ_0 is to select a value for x_0 and to use, for example, direct shooting to obtain J^{opt}. Then one element of x_0 is perturbed and a new value of J^{opt} is obtained. Finally, λ_0 is calculated as

$$\lambda_{k_0} = \frac{\Delta J^{\text{opt}}}{\Delta x_{k_0}} \tag{33}$$

A complete discussion of this method for the general optimal control problem is presented in [7].

IX. Adjoint-Control Transformation

The adjoint-control transformation is simply the solution of the optimality conditions for λ_0 in terms of physical quantities for which the values can be guessed [8]. For example, consider the launch of a rocket over a flat moon with no condition imposed on the final downrange. The steering angle for minimum final time satisfies the relation

$$\tan\theta = -\frac{\lambda_2}{\lambda_3}t + \frac{C}{\lambda_3} \tag{34}$$

where λ_2, λ_3, and C are three constants to be determined. From this relation, it is seen that

$$\frac{C}{\lambda_3} = [\tan\theta]_{t=0}, \qquad -\frac{\lambda_2}{\lambda_3} = \left[\frac{\mathrm{d}(\tan\theta)}{\mathrm{d}t}\right]_{t=0} \tag{35}$$

Hence, it is possible to relate λ_2/C and λ_3/C to the steering angle $\tan\theta$ and its derivative at $t = 0$. Reasonable guesses can be made for these quantities. A third equation comes from $H_f = -1$.

X. Conclusions

An often-used numerical method for solving optimal control problems is the shooting method. However, to make it work, a good guess is needed for the initial Lagrange multipliers. This Note concisely summarizes many of the available options. A standard optimal control problem without control bounds and its optimality conditions have been stated. Following a brief derivation of the shooting method, several methods for obtaining initial Lagrange multipliers have been discussed: parameterization of the control within the shooting method, direct shooting, collocation, pseudospectral method, dynamic programming, and adjoint-control transformation. Because the intent has been to use the shooting method, gradient information (which is not usually available for direct methods) can be used to improve direct shooting and collocation.

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